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Color-to-spin ribbon Schensted algorithms

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Abstract

A new Schensted bijection is given from colored permutations to pairs of standard k-ribbon tableaux, such that twice the total color of the colored permutation, is equal to the sum of the spins of the pair of tableaux. A highly nontrivial extension of this bijection is also given, from colored words to a pair of k-ribbon tableaux, one semistandard and the other standard. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

In [14] Stanton and White defined a bijection between k-colored permutations and pairs of standard k-ribbon (also called k-rim hook) tableaux of the same shape. This bijection is transported to a k-fold product of ordinary Schensted bijections by Littlewood's k-quotient bijection. A formulation of Stanton and White's bijection using chains of partitions was given by Fomin and Stanton [5].

The goal of this paper is to define a different bijection from colored permutations to pairs of standard k-ribbon tableaux of the same shape, such that twice the total color of a colored permutation, equals the sum of the spins of the corresponding pair of ribbon tableaux (Theorem 2). This new bijection has the involution property: taking a suitable inverse of the colored permutation has the effect of exchanging the two tableaux (Theorem 4). The color-to-spin ribbon Schensted Algorithm is introduced in a somewhat traditional manner, and is also formulated in Fomin's poset-theoretic framework [3–5]. A highly nontrivial semistandard extension of the color-to-spin bijection is also given (Theorem 5).

In the domino case ($k=2$) there is yet another such bijection due to Barbasch and Vogan [1] and Garfinkle [6] that is different from the domino ($k=2$) special cases of

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both of the above bijections. Garfinkle's recursive definition for the algorithm has been translated into the language of chains of partitions by van Leeuwen [15], who extended the algorithm to handle the case of nonempty 2-core. Surprisingly, it appears to be a new observation, that the Barbasch–Vogan–Garfinkle domino Schensted bijection also preserves spin [11]. This fact leads to a new formula for the q-analogue $c^{\lambda}_{\mu,\nu}(q)$ of the Littlewood–Richardson coefficients defined by Carré and Leclerc [2], who had already given a combinatorial description of the q-LR coefficients using Yamanouchi domino tableaux with spin. In [11] the new rule for the q-LR coefficients is used to derive an explicit formula in the case that μ and ν are rectangles; currently few explicit formulas exist for these polynomials.

More generally, by counting semistandard k-ribbon tableaux by content and spin, Lascoux, Leclerc, and Thibon defined a q-analogue $c^{\lambda}_{\mu^1,\dots,\mu^k}(q)$ of the multiplicity of the Schur function s_λ in a product $s_{\mu^1}s_{\mu^2}\dots s_{\mu^k}$ of Schur functions [8]. It has been shown by Leclerc and Thibon that these q-analogues are certain parabolic Kazhdan–Lusztig polynomials of affine type A [9]. Kashiwara and Tanisaki [7] have shown that these polynomials have nonnegative integer coefficients.

All of the above k-ribbon Schensted bijections reduce to the ordinary Schensted bijection when $k=1$.

The Stanton–White and the color-to-spin k-ribbon Schensted bijections generally produce different shaped tableaux for the same colored permutation. Even more interesting is the contrast between the "semistandard" extensions of these two bijections. The Stanton–White bijection generalizes more or less trivially as does Schensted's original bijection, using an obvious standardization relabeling. However the color-to-spin bijection has a nontrivial semistandard extension that is inherently incompatible with the usual standardization, but which retains the color-to-spin property (see Section 5).

The paper is organized as follows. Section 2 contains definitions and statements of the main results. Section 3 gives some new properties of the k-ribbon lattice. Section 4 defines the color-to-spin ribbon bijection in a somewhat traditional manner. Section 5 describes the semistandard extension of the color-to-spin bijection. Section 6 gives a poset-theoretic definition of the color-to-spin bijection. Section 7 contains remarks and comparisons with related algorithms. The appendix contains proofs of some technical lemmas.

The authors hope that this color-to-spin k-ribbon Schensted bijection will prove to be a useful tool for studying the q-LR coefficients of Lascoux et al. The authors thank one of the referees for suggesting a shorter and clearer proof of Lemma 6.

2. Definitions and results

2.1. Definitions

A *cell* is an ordered pair (i,j) of positive integers. The English (matrix-style) convention is adopted here: (i,j) represents the position in the ith row from the top and the jth column from the left. The (northwest to southeast) *diagonal* of the cell (i,j) is defined to be $\mathrm{diag}(i,j)=j-i$. Two partial orders on cells are used, one that depends on the diagonal index and the other on the column index. Say that the cell s is *southwest* (resp. strictly southwest) of the cell s' and write $s \leqslant_d s'$ (resp $s <_d s'$) if $\mathrm{diag}(s) \leqslant \mathrm{diag}(s')$ (resp. $\mathrm{diag}(s) < \mathrm{diag}(s')$). Write $s \leqslant_c s'$ (resp. $s <_c s'$) and say s is west (resp. strictly west) of s' if the column index of s is less than or equal to (resp. less than) that of s'.

A *partition* $\lambda=(\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ is a finite weakly decreasing sequence of positive integers. The *Ferrers diagram* or *shape* of λ is the set of cells $\{(i,j)\colon 1 \leqslant j \leqslant \lambda_i\}$, viewed as a left-justified array of matrix positions, with λ_i in the ith row. By abuse of notation, λ stands both for the partition itself and for its shape. Write $\mu \subset \lambda$ if the diagram of μ is a subset of that of λ. The set of partitions under the partial order $\subset$ is called Young's lattice $\mathbb{Y}$. If $\mu \subset \lambda$, denote by λ/μ the set difference of the diagram of λ minus the diagram of μ; such a set of cells is called a skew shape. For the skew shapes D and E, say that *E extends D* (or that *D is extended by E*) if there exist partitions $\nu \subset \mu \subset \lambda$ such that $D=\mu/\nu$ and $E=\lambda/\mu$. In this case let $D \cup E=\lambda/\nu$.

Let k be a positive integer which shall be fixed from now on. A *k-ribbon* is a connected skew shape consisting of k cells, at most one on each diagonal. From now on, all ribbons shall be k-ribbons. The *head* (written $\mathrm{hd}(h)$) of the ribbon h is its northeastmost cell. The tail (written $\mathrm{tl}(h)$) of h is its southwestmost cell. For two ribbons h and h', write $h \leqslant_c h'$ (resp. $h <_c h'$) and say that h is (resp. strictly) west of h' if the same is true of $\mathrm{hd}(h)$ and $\mathrm{hd}(h')$. Make similar definitions for $\leqslant_d$, $<_d$, and southwestness.

The spin of h (written $\mathrm{sp}(h)$) is the row index of $\mathrm{tl}(h)$ minus the row index of $\mathrm{hd}(h)$. Clearly for a k-ribbon h, $\mathrm{sp}(h) \in \{0,1,2,\ldots,k-1\}$.

Define the relation $\mu \lessdot \lambda$ on $\mathbb{Y}$ to mean that $\mu \subset \lambda$ and the skew shape λ/μ is a ribbon h. In this situation h is said to be *μ-addable* and *λ-removable*. The relation $\lessdot$ is the covering relation for a partial order on $\mathbb{Y}$. Each component of this poset has a unique minimum. These elements are called k-cores. The k-rim hook (or k-ribbon) lattice RH_k is the component of the empty shape $\varnothing$. Write $\mu \lessdot\!= \lambda$ to mean that either $\mu \lessdot \lambda$ or $\mu=\lambda$. On the set of μ-addable ribbons, $<_d$ and $<_c$ give the same total order.

In some situations, when working with a skew shape λ/μ, it is necessary to know the shapes λ and μ, not just their set difference. However, when this is the case the partitions μ and λ are implicitly or explicitly specified by context. For example, when it matters, a ribbon h appears in the context of being μ-addable or λ-removable, either of which specifies both λ and μ.

Let n be a fixed positive integer. A (standard) ribbon tableau of shape λ/μ is a chain of partitions of the form $\mu=\lambda^{(0)} \lessdot\!= \lambda^{(1)} \lessdot\!= \cdots \lessdot\!= \lambda^{(n)}=\lambda$. This ribbon tableau is said to contain i if $\lambda^{(i-1)} \lessdot \lambda^{(i)}$. A ribbon tableau can be depicted by placing every number i that it contains, in the corresponding ribbon $\lambda^{(i)}/\lambda^{(i-1)}$. The spin of a ribbon tableau is the sum of spins of its ribbons. A *horizontal ribbon strip* is a skew shape λ/μ that admits a ribbon tableau S of shape λ/μ in which the head of every ribbon of S is in the northmost cell in its column in λ/μ, such that the heads of the ribbons, proceed from west to east. If such a tableau exists it is necessarily unique, assuming that none of the shapes $\lambda^{(i)}/\lambda^{(i-1)}$ is empty for $1 \leqslant i \leqslant m$ where $\lambda = \lambda^{(m)}$. Call the sequence of ribbons given by S, the *standard ribbon tiling* (or standard tiling for short) of the horizontal ribbon strip λ/μ. The spin of a horizontal ribbon strip is defined to be the spin of the tableau that gives its standard tiling. Write $\mu \preccurlyeq \lambda$ if λ/μ is a horizontal ribbon strip. The following lemma, which is easy to verify, gives a local criterion for when a skew shape tiled by successive ribbons, is a horizontal ribbon strip. For a non-empty horizontal ribbon strip λ/μ, let $\mathbf{eastr}(\lambda/\mu)$ be the eastmost ribbon in the standard tiling of λ/μ.

Lemma 1. *Let $\mu \subset \lambda \lessdot \rho$. Then $\mu \prec \rho$ if and only if $\mu \preccurlyeq \lambda$, and either $\mu = \lambda$ or $\mathbf{eastr}(\lambda/\mu) <_c \rho/\lambda$. Moreover, if $\mu \prec \rho$ then the standard tiling of ρ/μ is given by the standard tiling of λ/μ together with ρ/λ, so that*

$$\mathrm{sp}(\rho/\mu) = \mathrm{sp}(\rho/\lambda) + \mathrm{sp}(\lambda/\mu). \tag{2.1}$$

Let T be a semistandard ribbon tableau of shape λ/μ. By definition T is a sequence of shapes $\mu = \lambda^{(0)} \preccurlyeq \lambda^{(1)} \preccurlyeq \cdots \preccurlyeq \lambda^{(n)}$, and is depicted by placing the letter i in each of the ribbons in the standard tiling of $\lambda^{(i)}/\lambda^{(i-1)}$ for all $1 \leqslant i \leqslant n$. The *content* of T is the sequence $(c_1, c_2, \ldots, c_n)$ where c_i is the number of ribbons in the standard ribbon tiling of $\lambda^{(i)}/\lambda^{(i-1)}$. The letter i is said to occur in T or T is said to contain the value i if $c_i > 0$. The spin of T (written $\mathrm{sp}(T)$) is the sum of the spins of its horizontal ribbon strips. The shape of T is written $\mathrm{sh}(T)$.

Fix a positive integer n. A *color* is an element of the set $\{0, 1, 2, \ldots, k-1\}$. A *value* is an element of the set $\{1, 2, \ldots, n\}$. A *place* is also an element of $\{1, 2, \ldots, n\}$. A *placed value* is an pair written $\binom{p}{v}$ where p is a place and v is a value. A *colored value* is a pair $\binom{c}{v}$ where c is a color and v a value. A *colored placed value* is an ordered triple (c, p, v) where c is a color, p is a place, and v is a value.

A *permutation* is a sequence of distinct values. An *indexed permutation* is a sequence of placed values such that the places strictly increase and the values are distinct. A *colored permutation* is a sequence of colored values whose values are distinct. A *colored indexed permutation* is a sequence of colored placed values that is an indexed permutation if its colors are ignored. The *total color* (denoted $\mathrm{tc}(\pi)$) of a colored (indexed) permutation π is the sum of the colors in its colored (placed) values. Let i be a place and j a value. For the k-colored indexed permutation π, denote by $\pi_{i,j}$ the subsequence of π given by removing all triples (c, p, v) such that $p > i$ or $v > j$.

Suppose S and T are semistandard ribbon tableaux such that $\mathrm{sh}(T)$ extends $\mathrm{sh}(S)$ and all the values that occur in S are strictly less than those that occur in T. If S and T are defined by the shape sequences $\mu^{(0)} \preccurlyeq \cdots \preccurlyeq \mu^{(n)}$ and $\lambda^{(0)} \preccurlyeq \cdots \preccurlyeq \lambda^{(n)}$ respectively, and S contains i as its maximum value, then let $S \cup T$ be the semistandard k-ribbon tableau defined by the sequence of shapes $\mu^{(0)} \preccurlyeq \cdots \preccurlyeq \mu^{(i)} \preccurlyeq \lambda^{(i+1)} \preccurlyeq \lambda^{(i+2)} \preccurlyeq \cdots \preccurlyeq \lambda^{(n)}$. In other words $S \cup T$ is obtained by putting together S and T.

We use the term "ribbon Schensted bijection" for any bijection of the form $\pi \mapsto (P(\pi), Q(\pi))$ from colored indexed permutations to pairs of k-ribbon tableaux of the same shape, satisfying the following properties:

(S1) The value v appears in π if and only if it does in $P(\pi)$.
(S2) The place p appears in π if and only if it does in $Q(\pi)$.
(S3) For all $1 \leqslant j \leqslant n$ and all $0 \leqslant i \leqslant n$, $P(\pi_{i,j})$ is obtained from $P(\pi_{i,j-1})$ by adjoining a ribbon labeled j if the value j appears in π. By definition if the value j does not appear in π then $\pi_{i,j} = \pi_{i,j-1}$, so that $P(\pi_{i,j}) = P(\pi_{i,j-1})$.
(S4) For all $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$, $Q(\pi_{i,j})$ is obtained from $Q(\pi_{i-1,j})$ by adjoining a ribbon labeled i if the place i appears in π. By definition if the place i does not appear in π then $\pi_{i,j} = \pi_{i-1,j}$ and $Q(\pi_{i,j}) = Q(\pi_{i-1,j})$.

The axioms (S1) and (S2) are content-preserving properties. Axiom (S3) says that the insertion of larger letters does not affect the positions of smaller letters. Axiom (S4) says that $Q(\pi)$ records the growth of the shape of $P(\pi)$ for left factors of π.

2.2. Results

Theorem 2. *There is a ribbon Schensted bijection $\pi \mapsto (P(\pi), Q(\pi))$ that satisfies the color-to-spin property*:

$$2\mathrm{tc}(\pi) = \mathrm{sp}(P(\pi)) + \mathrm{sp}(Q(\pi)). \tag{2.2}$$

We shall construct such a bijection which also satisfies the following involution property. Define the inverse π^{-1} of a colored indexed permutation π to be that given by exchanging the place and value in each colored placed value, and then resorting the new colored placed values so that their places increase.

Example 3. A colored indexed permutation π and its inverse are given below. The colored placed values are given by columns.

$$\pi = \begin{pmatrix} 0 & 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 4 & 2 \end{pmatrix}, \quad \pi^{-1} = \begin{pmatrix} 3 & 1 & 2 & 2 & 0 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix}.$$

Theorem 4. *The color-to-spin ribbon Schensted bijection has the following involution property. For any colored indexed permutation π*,

$$P(\pi^{-1}) = Q(\pi),$$

$$Q(\pi^{-1}) = P(\pi).$$

There is a nontrivial semistandard extension of the bijection in Theorem 2. A *colored word* is a finite sequence of colored values. A *colored indexed word* is a finite sequence of colored placed values such that the places strictly increase. Given a colored word, there is a natural way to make it into a colored indexed word, namely, by giving the jth colored value the place j.

Theorem 5. *There is a bijection between colored indexed words π and pairs of ribbon tableaux (P, Q) of the same shape, where P is semistandard and Q is standard, also denoted by $P = P(\pi)$ and $Q = Q(\pi)$, such that:*

1. *The number of occurrences of the value v in P and in π are the same.*
2. *The place p appears in π if and only if it does in Q.*
3. (2.2) *holds.*

3. The k-ribbon lattice

Some new observations on the k-ribbon lattice are required to define the color-to-spin k-ribbon Schensted bijection.

3.1. The k-ribbon lattice and spin

Lemma 6. *Let μ be a shape and c a color.*

1. *There is a μ-addable ribbon of spin c.*
2. *For any μ-removable ribbon h with $\mathrm{sp}(h) \leqslant c$, there is a μ-addable ribbon of spin c that is strictly southwest of h.*

The proof of Lemma 6 is deferred to Section 8.

Let μ be a partition and c a color. Define **firstr**(μ, c) ("first ribbon") to be the northeastmost μ-addable ribbon of spin c; it exists by Lemma 6.

Let h be a μ-removable ribbon of spin c. Define **nextr**(μ, h) ("next ribbon") to be the northeastmost μ-addable ribbon of spin c that is strictly southwest of h; this also exists by Lemma 6.

The reader is warned that the positions of the ribbons given by **firstr** and **nextr** can be far to the southwest; **firstr** need not be close to the northeast part of the shape, and **nextr** need not be close to the ribbon h, both of which are the case for the "row insertion" version of the Stanton–White k-ribbon Schensted bijection.

Let λ be a partition and h a λ-removable ribbon. Define **prevr**(λ, h) ("previous ribbon") to be the southeastmost $(\lambda - h)$-removable ribbon of the same spin as h that lies strictly to the northeast of h. Such a ribbon need not exist; in this case define **prevr**$(\lambda, h) = \emptyset$. Note that **prevr** is the inverse of **firstr** or **nextr** according as its value is empty or not.

Proposition 7. 1. *Let μ be a partition, c a color and $h = \mathbf{firstr}(\mu, c)$. Then $\mathbf{prevr}(\mu \cup h, h) = \emptyset$. Conversely, let λ be a partition and h a λ-removable ribbon such that $\mathbf{prevr}(\lambda, h)$ is empty. Then $h = \mathbf{firstr}(\lambda - h, \mathrm{sp}(h))$.*

2. *Let μ be a partition, h a μ-removable ribbon, and $h' = \mathbf{nextr}(\mu, h)$. Then $h = \mathbf{prevr}(\mu \cup h', h')$. Conversely, let λ be a partition and h' a λ-removable ribbon such that $h = \mathbf{prevr}(\lambda, h')$ is nonempty. Then $h' = \mathbf{nextr}(\lambda - h', h)$.*

3.2. Bumpout and Bumpin

We recall the operation of **bumpout** defined in [16]. Let h_1 and h_2 be ribbons. Define the set of cells

$$\mathbf{bumpout}(h_1, h_2) = (h_2 \setminus h_1) \cup \{(i+1, j+1) \mid (i,j) \in h_1 \cap h_2\}.$$

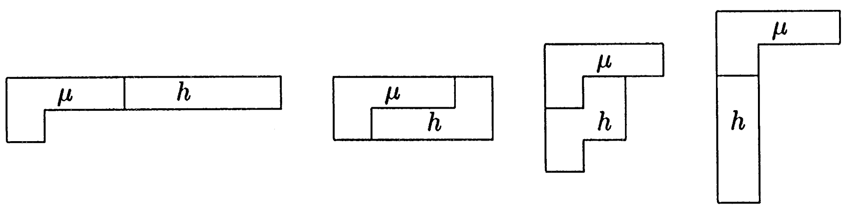
By definition,

$$\begin{aligned}
&\mathrm{diag}(\mathrm{hd}(\mathbf{bumpout}(h_1, h_2))) = \mathrm{diag}(\mathrm{hd}(h_2)) \\
&\mathrm{diag}(\mathrm{tl}(\mathbf{bumpout}(h_1, h_2))) = \mathrm{diag}(\mathrm{tl}(h_2)) \\
&\mathrm{hd}(\mathbf{bumpout}(h_1, h_2)) = \mathrm{hd}(h_2) \quad \text{if } h_1 <_d h_2 \\
&\mathrm{tl}(\mathbf{bumpout}(h_1, h_2)) = \mathrm{tl}(h_2) \quad \text{if } h_2 <_d h_1.
\end{aligned} \tag{3.1}$$

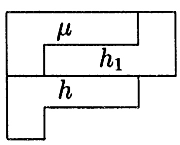
Proposition 8. *Let $\mu \lessdot v^i$ for $i = 1, 2$ with $v^1 \neq v^2$. Then there is a unique partition λ such that $v^i \lessdot \lambda$ for $i = 1, 2$. Writing $h_i = v^i/\mu$ for $i = 1, 2$, one has $\mathbf{bumpout}(h_1, h_2) = \lambda/v^1$ and $\mathbf{bumpout}(h_2, h_1) = \lambda/v^2$. Moreover*

$$\mathrm{sp}(h_1) + \mathrm{sp}(h_2) = \mathrm{sp}(\mathbf{bumpout}(h_1, h_2)) + \mathrm{sp}(\mathbf{bumpout}(h_2, h_1)) \tag{3.2}$$

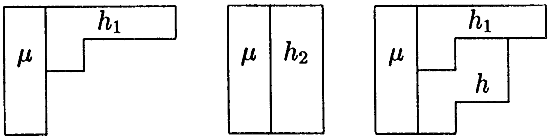
Example 9. Let $k = 4$ and $\mu = (3, 1)$. For $c \in \{0, 1, 2, 3\}$ the μ-addable ribbons $h = \mathbf{firstr}(\mu, c)$ are given in order below.



With $\mu = (4, 4)$ and $h_1 = (4, 4)/(3, 1)$, the ribbon $h = \mathbf{nextr}(\mu, h_1)$ is given below.



Finally, let $\mu=(1,1,1,1)$ and let h_1 and h_2 be the μ-addable ribbons indicated below. Then $h=\mathbf{bumpout}(h_1,h_2)$ is indicated below.



Let h_1' and h_2' be ribbons. Define

$$\mathbf{bumpin}(h_1',h_2')=(h_1'\setminus h_2')\cup\{(i-1,j-1)\mid(i,j)\in h_1'\cap h_2'\}.$$

Proposition 10. *Let λ be a partition and h_1' and h_2' distinct λ-removable k-ribbons. Then* $\mathbf{bumpin}(h_1',h_2')$ *is a $(\lambda-h_2')$-removable ribbon. Moreover*

$$h_1=\mathbf{bumpin}(h_1',h_2') \quad \textit{and} \quad h_2=\mathbf{bumpin}(h_2',h_1')$$

if and only if

$$\mathbf{bumpout}(h_2,h_1)=h_1' \quad \textit{and} \quad \mathbf{bumpout}(h_1,h_2)=h_2'.$$

4. Traditional description of color-to-spin bijection

In this section, the color-to-spin ribbon Schensted bijection is described in traditional terms.

4.1. Insertion of a colored value into a ribbon tableau

Let T be a (standard) ribbon tableau that does not contain the value v. The insertion of the colored value $\binom{c}{v}$ into T is the following algorithm, which produces the tableau P. This is denoted $P=T\overset{cs}{\leftarrow}\binom{c}{v}$. Say that $v_1<v_2<\cdots<v_m$ are the values in T that are greater than v, and h_j the shapes of their ribbons. Let $v=v_0$ and let T_j be obtained from T by removing all the values that are greater than v_j for $0\leqslant j\leqslant m$.

1. Remove from T all ribbons of value greater than v, leaving T_0.
2. Let P_0 be obtained by adjoining to T_0, the ribbon $h_0=\mathbf{firstr}(\mathrm{sh}(T_0),c)$ containing the single value v.
3. Given P_{j-1}, let $h_{j-1}'=\mathrm{sh}(P_{j-1})/\mathrm{sh}(T_{j-1})$. P_j is obtained from P_{j-1} by adjoining a ribbon of value v_j, at the position given by

$$\begin{cases} h_j & \text{if } h_{j-1}'\cap h_j=\varnothing,\\ \mathbf{nextr}(\mathrm{sh}(P_{j-1}),h_j) & \text{if } h_{j-1}'=h_j,\\ \mathbf{bumpout}(h_{j-1}',h_j) & \text{otherwise.}\end{cases} \tag{4.1}$$

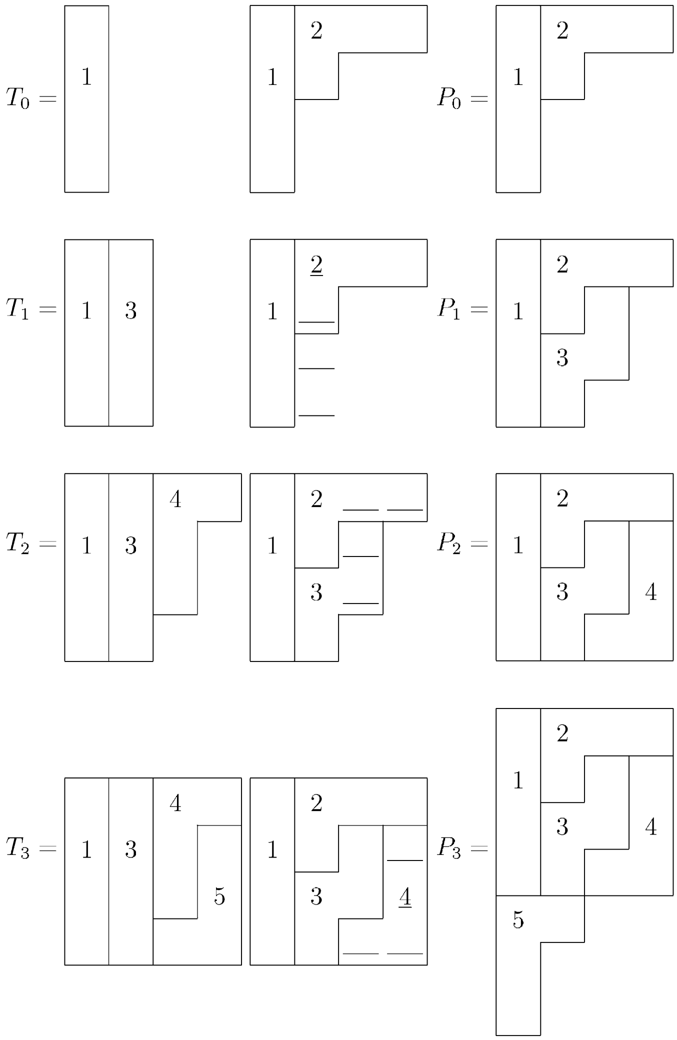
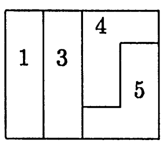


Fig. 1.

Let $P = P_m$ and $h' = \mathrm{sh}(P)/\mathrm{sh}(T)$. Then

$$\mathrm{sp}(P) + \mathrm{sp}(h') = \mathrm{sp}(T) + 2c. \tag{4.2}$$

Example 11. $P = (T \overset{cs}{\leftarrow} (\begin{smallmatrix}1\\2\end{smallmatrix}))$ is computed in Fig. 1. $v_0 = 2, v_1 = 3, v_2 = 4, v_3 = 5$. The underlined positions indicate the ribbon h_j.

$T =$ 

P_0 is obtained from T_0 by a **firstr**. P_1 is obtained from T_1 by a **bumpout** of the ribbon of value 3. P_2 and P_3 are obtained from T_2 and T_3 by **nextr**. In this example h' is the ribbon containing the value 5 in $P = P_3$. Checking the spin, $\mathrm{sp}(P) + \mathrm{sp}(h') = (3 + 1 + 2 + 2 + 2) + 2 = 12$ and $\mathrm{sp}(T) + 2c = (3 + 3 + 2 + 2) + 2 \times 1 = 12$.

4.2. *Insertion of a colored indexed permutation*

Consider a colored indexed permutation π with its colored placed values written in columns.

$$\pi = \begin{pmatrix} c_1 & c_2 & \cdots & c_L \\ p_1 & p_2 & \cdots & p_L \\ v_1 & v_2 & \cdots & v_L \end{pmatrix}.$$

Let $P_0 = Q_0 = \emptyset$. For $1 \leqslant j \leqslant L$, let $P_j = (P_{j-1} \overset{cs}{\leftarrow} \binom{c_j}{v_j})$ and let Q_j be obtained from Q_{j-1} by adjoining a ribbon $\mathrm{sh}(P_j)/\mathrm{sh}(P_{j-1})$ which is labeled p_j. Let $P = P_L$ and $Q = Q_L$. Define

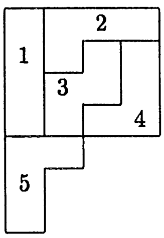
$$P = P(\pi) = (\varnothing \overset{cs}{\leftarrow} \pi)$$

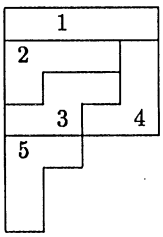
$$Q = Q(\pi).$$

Example 12. Consider the colored permutation

$$\pi = \begin{pmatrix} 0 & 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 4 & 2 \end{pmatrix}$$

whose colored placed values are written in columns. Its total color is $0+2+3+2+1 = 8$. The tableaux $P = P(\pi)$ and $Q = Q(\pi)$ are given by

$P =$ 

$Q =$ 

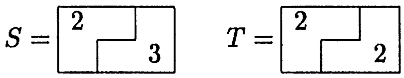
So $\mathrm{sp}(P) + \mathrm{sp}(Q) = 10 + 6 = 16$ which agrees with $2\mathrm{tc}(\pi) = 2 \times 8$.

5. Semistandard color-to-spin bijection

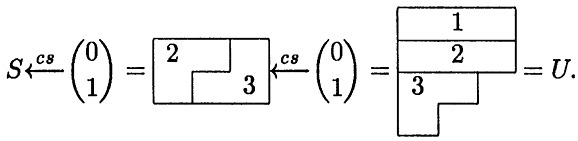
Schensted [10] gave a natural extension of his insertion algorithm for permutations and standard tableaux, to allow for arbitrary words and semistandard tableaux. This extension is defined using a simple relabeling process known as *standardization*. The Stanton–White k-ribbon insertion has a similar extension to colored words with repeated values and semistandard ribbon tableaux using a simple relabeling; within each of the horizontal ribbon strips defining a semistandard ribbon tableau, the ribbons of the standard tiling are labeled in increasing order from left to right.

One cannot define a semistandard extension of the color-to-spin ribbon bijection in the same way. Suppose one has a semistandard extension that retains the color-to-spin property, respects restriction to small values, and is defined by iterating the insertion of single colored values. These are essentially the axioms (S1)–(S4). Then such an extension cannot be compatible with the usual standardization of a semistandard ribbon tableau given by labeling the standard tiling of a given horizontal ribbon strip from left to right with increasing values. This is seen by the following counterexample.

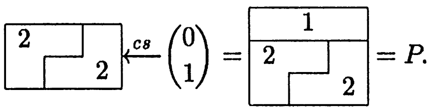
Example 13. Let $k=3$. Consider the ribbon tableau S, which is the standardization in the alphabet $\{2,3\}$, of the semistandard ribbon tableau T.



Now the smaller colored value $\binom{0}{1}$ is inserted into S.



It will be shown that



There is no choice about the ribbon of value 1 in P due to the color-to-spin property and the assumption that larger values do not affect the insertion of smaller values. Now P must be a ribbon tableau with a single 1 and two 2's, of a shape obtained from that of T by adjoining a ribbon h'. Moreover it must satisfy $\mathrm{sp}(P)-\mathrm{sp}(T)+\mathrm{sp}(h')=2c=0$ by (4.2). Under these constraints the only such ribbon tableau P is the one given above. Note that $\mathrm{sh}(P)\neq\mathrm{sh}(U)$ despite the fact that S is the standardization of T (in the appropriate alphabet) and 1 is smaller than all other letters that appear.

To specify any kind of insertion that respects restriction to small values, it is sufficient to define two kinds of operations. As an example, consider Schensted's row insertion operation $P=(T \leftarrow v)$ of a value v into the semistandard tableau T with result P. This insertion cannot move letters smaller than v so the positions of such letters in P are the same as they are in T. The letters larger than v do not affect the positions of the v's. To determine the positions of the v's in P one has to define the row insertion of v, into a skew semistandard tableau consisting only of v's. This is a special case of *skew insertion* [12]. It remains to determine the positions of the values $x > v$. Suppose by induction the positions of the $(x-1)$'s have been determined already. By induction the subshapes of T and P containing values strictly less than x, differ at most by adding a cell. This cell may intrude into the subtableau of x's, so the subtableau of x's needs to be modified. This modification process is a special case of *internal insertion* [12].

For the semistandard extension of the color-to-spin insertion, the analogue of the horizontal strip of v's is the horizontal ribbon strip containing the values v and the analogue of a cell is a ribbon. It is enough to define two operations: the skew insertion of the spin c into a horizontal ribbon strip, and the internal insertion on a horizontal ribbon strip λ/μ, at a μ-addable ribbon.

5.1. *More on* RH_k *and spin*

In this section, the semistandard analogues of **firstr** and **nextr** are defined. For this, more detailed properties of the ribbon lattice are required. The following Lemma is proved in Section 8.

Lemma 14. *Let c be a color and $\mu \leqslant \lambda$ with $\mu = \lambda^{(0)} \lessdot \lambda^{(1)} \lessdot \cdots \lessdot \lambda^{(m)} = \lambda$* and *$h_j = \lambda^{(j)}/\lambda^{(j-1)}$, such that h_1 through h_m form the standard tiling of λ/μ.*

(B1) *There is an index $0 \leqslant p \leqslant m$ and a $\lambda^{(p)}$-addable ribbon h' of spin c, where $\mu \prec \lambda^{(p)} \cup h'$.*

(B2) *Suppose h is a λ-removable ribbon with* $\mathrm{sp}(h) \leqslant c$, *such that $h_m <_d h$ if $m > 0$. Then there is a p and h' as in* (B1), *such that $h' <_d h$.*

(B3) *In each of* (B1) *and* (B2), *take p maximal and then h' northeastmost. With these choices, if $p < m$ then $h' <_d h_{p+1}$.*

Lemma 14 is false without the hypothesis on h_m. (B3) is crucial for the properties of the semistandard color-to-spin algorithm.

5.2. *Bumpout of a horizontal ribbon strip*

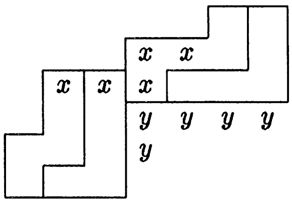
Let $\mu \leqslant \lambda$ with $\mu = \lambda^{(0)} \lessdot \lambda^{(1)} \lessdot \cdots \lessdot \lambda^{(m)} = \lambda$ with $h_j = \lambda^{(j)}/\lambda^{(j-1)}$, such that the h_j give the standard tiling of the horizontal ribbon strip λ/μ. Let $\mu \lessdot v$ and $h = v/\mu$. Suppose further that $\mathrm{hd}(h) \neq \mathrm{hd}(h_j)$ for all j. Then the bumpout of the horizontal ribbon strip λ/μ by the ribbon h is defined as follows. Start with the partition v. Adjoin ribbons h_1 through h_m using the rules specified in (4.1), resulting in shapes $v = \rho^{(0)} \lessdot \rho^{(1)} \lessdot \cdots \lessdot \rho^{(m)} = \rho$, where $h'_j = \rho^{(j)}/\lambda^{(j)}$. Write $\mathrm{sbumpout}(\mu, v, \lambda) = (v, \lambda, \rho)$. One should imagine that the horizontal ribbon strip λ/μ consists of v's, has been displaced by values smaller than v at h, and adjusts itself to settle in position ρ/v, causing a displacement of values larger than v at the ribbon $h' = \rho/\lambda$. The key feature of this algorithm is that the **nextr** case of (4.1) does not occur; this is clear since

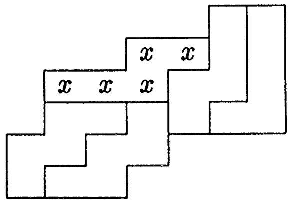
bumpout preserves the diagonal of the head of any ribbon, and all heads of ribbons start on different diagonals by assumption.

Proposition 15. *Let* sbumpout$(\mu,\nu,\lambda)=(\nu,\lambda,\rho)$. *Then* ρ/ν *is a horizontal ribbon strip with standard tiling given by* $\rho^{(j)}/\rho^{(j-1)}$. *Moreover*

$$\mathrm{sp}(\rho/\nu)+\mathrm{sp}(\rho/\lambda)=\mathrm{sp}(\lambda/\mu)+\mathrm{sp}(\nu/\mu). \tag{5.1}$$

Example 16. Let $k=5$. A horizontal ribbon strip λ/μ, its standard tiling, a ribbon $h=\nu/\mu$ (whose cells are indicated by an x) and the resulting horizontal ribbon strip ρ/ν (the ribbons outside h) are given. The ribbon $h'=\rho/\lambda$ is indicated by y's.

$(\lambda/\mu)\cup h' =$ 

$h\cup(\rho/\nu) =$ 

Checking spins, $\mathrm{sp}(\rho/\nu)+\mathrm{sp}(\rho/\lambda)=(2+2+3+3)+1$ and $\mathrm{sp}(\lambda/\mu)+\mathrm{sp}(\nu/\mu)=(3+3+2+2)+1$.

5.3. Internal insertion

Let $\mu\preccurlyeq\lambda$ with the notation for the standard tiling of λ/μ as in Section 5.2. Let $\mu\lessdot\nu$ and $h=\nu/\mu$. The internal insertion on the horizontal ribbon strip λ/μ at h, shall be denoted **iins**$(\mu,\nu,\lambda)=(\nu,\lambda,\rho)$; it satisfies $\nu\preccurlyeq\rho$, $\lambda\lessdot\rho$, and (5.1).

Suppose that $\mathrm{hd}(h)\neq\mathrm{hd}(h_j)$ for all j. Let **iins**$(\mu,\nu,\lambda)=$ sbumpout(μ,ν,λ); this has the desired properties. Otherwise suppose $h=h_x$. Observe that

$$\mathrm{sbumpout}(\mu,\nu,\lambda^{(x-1)})=(\nu,\lambda^{(x-1)},\lambda^{(x)}). \tag{5.2}$$

This shows that λ/ν is a horizontal ribbon strip whose standard tiling is obtained by combining those of $\lambda^{(x)}/\nu$ and $\lambda/\lambda^{(x)}$. By (5.1),

$$\mathrm{sp}(\lambda^{(x)}/\nu)+\mathrm{sp}(h_x)=\mathrm{sp}(\lambda^{(x-1)}/\mu)+\mathrm{sp}(\nu/\mu). \tag{5.3}$$

Adding $\mathrm{sp}(h_x)+\mathrm{sp}(\lambda/\lambda^{(x)})$, one obtains

$$\mathrm{sp}(\lambda/\nu)+2\,\mathrm{sp}(h_x)=\mathrm{sp}(\lambda/\mu)+\mathrm{sp}(\nu/\mu). \tag{5.4}$$

Let $\nu=\rho^{(0)}\lessdot\rho^{(1)}\lessdot\cdots\lessdot\rho^{(x-1)}=\lambda^{(x)}$ and $h'_j=\rho^{(j)}/\rho^{(j-1)}$ for $1\leqslant j\leqslant x-1$, such that the h'_j give the standard tiling of $\lambda^{(x)}/\nu$. Let p and h' be as in Lemma 14 (B2) for $\lambda^{(x)}/\nu$, h_x, and $c=\mathrm{sp}(h_x)$. By Lemma 14(B3), $h'<_d h'_{p+1}$. Therefore, it makes sense to define ρ by sbumpout$(\rho^{(p)},\rho^{(p)}\cup h',\lambda)=(\rho^{(p)}\cup h',\lambda,\rho)$. ρ/ν is a horizontal ribbon strip, whose standard tiling consists of those of $\rho^{(p)}/\nu$, h', and $\rho/(\rho^{(p)}\cup h')$. Again by (5.1),

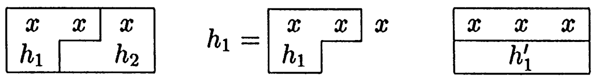
$$\mathrm{sp}(\rho/(\rho^{(p)}\cup h'))+\mathrm{sp}(\rho/\lambda)=\mathrm{sp}(\lambda/\rho^{(p)})+\mathrm{sp}(h'). \tag{5.5}$$

Recalling that $\mathrm{sp}(h') = c = \mathrm{sp}(h_x)$ and adding $c + \mathrm{sp}(\rho^{(p)}/v)$,

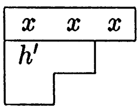
$$\mathrm{sp}(\rho/v) + \mathrm{sp}(\rho/\lambda) = \mathrm{sp}(\lambda/v) + 2c, \tag{5.6}$$

which, together with (5.4), yields (5.1).

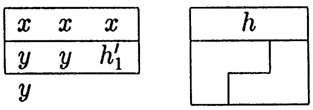
Example 17. Take the situation in Example 13. The insertion of the colored value $\binom{0}{1}$ into T creates an internal insertion on the horizontal ribbon strip $(3,3)/()$ occupied by the 2's, at the ribbon $h = (3)$. That is, $\mu = ()$, $v = (3)$, and $\lambda = (3,3)$, and we wish to compute $\mathbf{iins}(\mu, v, \lambda) = (v, \lambda, \rho)$. The standard tiling of $(3,3)/()$ is indicated by h_1 and h_2 and the cells of the ribbon h by the letter x. We have $\mathrm{hd}(h) = \mathrm{hd}(h_2) = (1,3)$ so $x = 2$. The operation (5.2) results in the standard tiling of $\lambda^{(x)}/v = (3,3)/(3)$ given by h'_1.



Applying Lemma 14(B2) to $(3,3)/(3)$ and the ribbon h_2, we have $p = 0$, and h' is given by



Finally, h' bumps out the strip consisting of h'_1. The cells of h' are marked with the letter y.



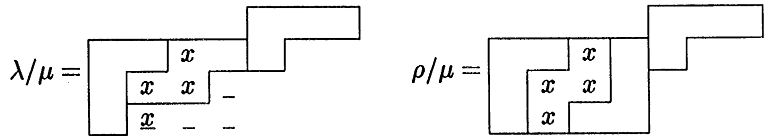
5.4. Skew insertion

Let $\mu \preccurlyeq \lambda$ with notation for the standard tiling of λ/μ as in Section 5.2. The skew insertion of the spin c into λ/μ is denoted by $\mathbf{skins}(\mu, \lambda, c) = (\mu, \lambda, \rho)$, and satisfies $\lambda \lessdot \rho$, $\mu \prec \rho$, and

$$\mathrm{sp}(\rho/\mu) + \mathrm{sp}(\rho/\lambda) = \mathrm{sp}(\lambda/\mu) + 2c. \tag{5.7}$$

Let p and h' be as in Lemma 14(B1) applied to λ/μ and c. By (B3) $h' <_d h_{p+1}$. Define ρ by $\mathrm{sbumpout}(\lambda^{(p)}, \lambda^{(p)} \cup h', \lambda) = (\lambda^{(p)} \cup h', \lambda, \rho)$. As before ρ/μ is a horizontal ribbon strip whose standard tiling consists of those of $\lambda^{(p)}/\mu$, h', and $\rho/(\lambda^{(p)} \cup h')$. As before, (5.7) follows from (5.1).

Example 18. Let $k = 4$, $c = 2$, and λ/μ be the horizontal ribbon strip with standard tiling given below. Then $p = 1$ and the skew insertion of c into λ/μ is computed below. The cells of the ribbon h' are indicated by an x. The cells of the ribbon ρ/λ are underlined.



Checking spins, $\mathrm{sp}(\rho/\mu) + \mathrm{sp}(\rho/\lambda) = (2+2+2+1)+1 = 8$ and $\mathrm{sp}(\lambda/\mu) + 2c = (2+1+1)+2\times 2 = 8$.

5.5. Insertion of a colored value into a semistandard ribbon tableau

Finally the insertion of a colored value $\binom{c}{v}$ into a semistandard ribbon tableau T, is defined; it shall be denoted $P = (T \overset{cs}{\longleftarrow} \binom{c}{v})$. Let T be defined by the sequence of shapes $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(n)} = \mathrm{sh}(T)$ where $\lambda^{(j)}/\lambda^{(j-1)}$ is the horizontal ribbon strip containing the values j. Define the semistandard ribbon tableau P given by the chain of shapes $\emptyset = \sigma^{(0)} \preccurlyeq \cdots \preccurlyeq \sigma^{(n)}$ and horizontal ribbon strips $\sigma^{(j)}/\sigma^{(j-1)}$ as follows. Let $\sigma^{(j)} = \lambda^{(j)}$ for $0 \leqslant j < v$. Define $\sigma^{(v)}$ by $\mathbf{skins}(\lambda^{(v-1)}, \lambda^{(v)}, c) = (\lambda^{(v-1)}, \lambda^{(v)}, \sigma^{(v)})$. For $v < j \leqslant n$, define $\sigma^{(j)}$ by $\mathbf{iins}(\lambda^{(j-1)}, \sigma^{(j-1)}, \lambda^{(j)}) = (\sigma^{(j-1)}, \lambda^{(j)}, \sigma^{(j)})$.

5.6. Insertion of a colored word

Let π be a colored indexed word of length L with colored placed values (c_j, p_j, v_j) for $1 \leqslant j \leqslant L$. Let P_0 be the empty tableau and $P_j = (P_{j-1} \overset{cs}{\longleftarrow} \binom{c_j}{v_j})$ and let Q_j be given by adjoining the place p_j at the ribbon $\mathrm{sh}(P_j)/\mathrm{sh}(P_{j-1})$ for $1 \leqslant j \leqslant L$. Let $P = P_L$ and $Q = Q_L$. Then P is a semistandard ribbon tableau and Q is a standard ribbon tableau of the same shape. Write $P = P(\pi) = (\emptyset \overset{cs}{\longleftarrow} \pi)$ and $Q(\pi) = Q$. Using the techniques developed for the standard ribbon tableau case, one may prove Theorem 5.

6. Poset-theoretic definitions

We discuss the general approach of Fomin [4] to Schensted-like bijections, applied to the case of k-ribbon Schensted bijections. There is an isomorphism $\mathrm{RH}_k \cong \mathbb{Y}^k$ [5]. It follows that RH_k is a k-differential poset in the language of [13] or a k-self dual graded graph in that of [3,4]. By [4, Lemma 3.4.2, Theorem 3.6.1] this guarantees the existence of a k-ribbon Schensted bijection. However, many such bijections exist, and it is not at all clear that there should be one that also satisfies the color-to-spin property (2.2). Section 6.1 discusses what must be common to the construction of all such bijections and then makes the particular choices which define the color-to-spin k-ribbon Schensted bijection.

6.1. k-correspondences for k-ribbon Schensted bijections

Given any k-ribbon Schensted bijection $\pi \mapsto (P(\pi), Q(\pi))$, one may associate a matrix $\tau(\pi)$ of partitions with entries

$$\tau_{i,j}(\pi) = \mathrm{sh}(P(\pi_{i,j})) \tag{6.1}$$

for $0 \leqslant i, j \leqslant n$. By (S1) and (S2), the nth row (resp. column) of this matrix gives the tableau $P(\pi)$ (resp. $Q(\pi)$). Define the matrix $c(\pi)$ with entries in the set $\{0, 1, \dots, k-1\} \cup \{\perp\}$ where $\perp$ is a special symbol, by

$$c_{i,j}(\pi) = \begin{cases} c & \text{if } \pi \text{ contains the triple } (c, i, j), \\ \perp & \text{otherwise.} \end{cases} \tag{6.2}$$

The properties (S1)–(S4) impose restrictions on the two-by-two submatrices in rows $i-1$ and i, and columns $j-1$ and j, for $1 \leqslant i, j \leqslant n$. The resulting rules for the two-by-two submatrix, are axiomatized in the definition of a k-correspondence [4]. Consider the following two-by-two matrix of partitions, equipped with arrows.

$$\begin{array}{ccc} \mu & \xrightarrow{h_2} & \nu^2 \\ {\scriptstyle h_1}\big\downarrow & & \big\downarrow{\scriptstyle h_1'} \\ \nu^1 & \xrightarrow[h_2']{} & \lambda \end{array} \tag{6.3}$$

In this diagram an arrow labeled h from a partition α to a partition β means that $\alpha \lessdot= \beta$ and $h = \beta/\alpha$. Informally, an inverse k-correspondence is a way to determine the south and east edges of the two-by-two matrix given the north and west ones, in a bijective manner so that the north and west edges may be found, given the south and east ones.

Let $\mathscr{A}$ be the set of 4-tuples (μ, ν^1, ν^2, c) where $\mu \lessdot= \nu^i$ for $i = 1, 2$ and c is either a k-color or the symbol $\perp$, and $c \neq \perp$ only if $\mu = \nu^1 = \nu^2$. Given such an element of $\mathscr{A}$ write

$$h_i = \nu^i/\mu \quad \text{for } i = 1, 2. \tag{6.4}$$

h_i is either empty or a μ-addable k-ribbon. Let $\mathscr{B}$ be the set of triples (ν^1, ν^2, λ) such that $\nu^i \lessdot= \lambda$ for $i = 1, 2$. Given such an element of $\mathscr{B}$, write

$$h_1' = \lambda/\nu^2 \quad h_2' = \lambda/\nu^1. \tag{6.5}$$

h_i' is either empty or a λ-removable k-ribbon.

An inverse k-correspondence is a bijection $\Psi : \mathscr{A} \to \mathscr{B}$ such that $\Psi(\mu, \nu^1, \nu^2, c)$ is of the form (ν^1, ν^2, λ) for some λ, where λ satisfies the conditions (I1)–(I3) below.

(I1) If $h_1 = h_2 = \varnothing$ and $c = \perp$ then $\lambda = \mu$.
(I2) If $h_1 \neq \varnothing$ and $h_2 = \varnothing$ then $\lambda = \nu^1$.
(I3) If $h_2 \neq \varnothing$ and $h_1 = \varnothing$ then $\lambda = \nu^2$.

It is easily seen that (I1)–(I3) are consequences of (S1)–(S4). In other words, a k-ribbon Schensted bijection gives rise to a k-correspondence that is defined in terms of the two-by-two submatrices of $\tau(\pi)$. Conversely, given a k-correspondence Ψ, the matrix $\tau(\pi)$ is recovered by starting with $\tau_{i,j}(\pi)=\varnothing$ for pairs (i,j) with $i=0$ or $j=0$, and then use the equation

$$\Psi(\tau_{i-1,j-1},\tau_{i,j-1},\tau_{i-1,j},c_{ij})=(\tau_{i,j-1},\tau_{i-1,j},\tau_{i,j}) \tag{6.6}$$

to inductively define $\tau_{i,j}$ in terms of matrix entries to the north and west. By [4, Lemma 3.5.8] the matrix $(\tau_{i,j})$ is independent of the order in which the entries are computed. The tableaux $P(\pi)$ and $Q(\pi)$ are defined respectively by the nth row and nth column of the matrix $\tau(\pi)$ that was just computed by Ψ. Thus a k-correspondence induces a k-ribbon Schensted bijection. This is the content of [4, Theorem 3.6.1]. The matrix $(\tau_{i,j})$ (resp. $(c_{i,j})$) is called a Φ-growth (resp. r-colored diagonal set) in [4] where $\Phi=\Psi^{-1}$.

In the case of RH_k, an inverse k-correspondence, which by definition has image $\mathscr{B}$, automatically satisfies the following property, by Proposition 8:

(I4) If h_1 and h_2 are nonempty and distinct, then $\lambda=v^1\cup\mathbf{bumpout}(h_1,h_2)$.

To define a particular inverse k-correspondence for RH_k, one must complete the remaining cases for h_1,h_2,c. The particular choices that induce the color-to-spin k-ribbon Schensted bijection, are given below.

(I5) If $h_1=h_2=\varnothing$ and $c\neq\perp$ then $\lambda=\mu\cup\mathbf{firstr}(\mu,c)$.
(I6) If $h_1=h_2\neq\varnothing$ then $\lambda=v^1\cup\mathbf{nextr}(v^1,h_1)$.

The following result is readily verified from the definitions.

Proposition 19. *With the above notation, the map $\Psi:\mathscr{A}\to\mathscr{B}$ defined by* (I1) *through* (I6) *is well-defined and satisfies*

$$\mathrm{sp}(h_1')+\mathrm{sp}(h_2')=\mathrm{sp}(h_1)+\mathrm{sp}(h_2)+\begin{cases}0 & \text{if } c=\perp,\\ 2c & \text{if } c\neq\perp.\end{cases} \tag{6.7}$$

$$\Psi(\mu,v^1,v^2,c)=\Psi(\mu,v^2,v^1,c). \tag{6.8}$$

To show that our map Ψ is an inverse k-correspondence, it remains to show it is bijective. Its inverse map Φ is defined as follows. Given $(v^1,v^2,\lambda)\in\mathscr{B}$, let h_1' and h_2' be defined by (6.5). Then $\Phi(v^1,v^2,\lambda)$ has the form (μ,v^1,v^2,c), $c=\perp$ except in case (C5) below, and μ is defined by:

(C1) If $h_1'=h_2'=\varnothing$ then $\mu=\lambda$.
(C2) If $h_1'\neq\varnothing$ and $h_2'=\varnothing$ then $\mu=v^2$.
(C3) If $h_2'\neq\varnothing$ and $h_1'=\varnothing$ then $\mu=v^1$.
(C4) If h_1' and h_2' are nonempty and distinct, then $\mu=v^1-\mathbf{bumpin}(h_1',h_2')$.
(C5) If $h_1'=h_2'\neq\varnothing$ and $\mathbf{prevr}(\lambda,h_1')=\varnothing$, then $\mu=v^1=v^2$ and $c=\mathrm{sp}(h_1')=\mathrm{sp}(h_2')$.
(C6) If $h_1'=h_2'\neq\varnothing$ and $\mathbf{prevr}(\lambda,h_1')\neq\varnothing$ then $\mu=v^1-\mathbf{prevr}(\lambda,h_1')$.

The following result is verified easily.

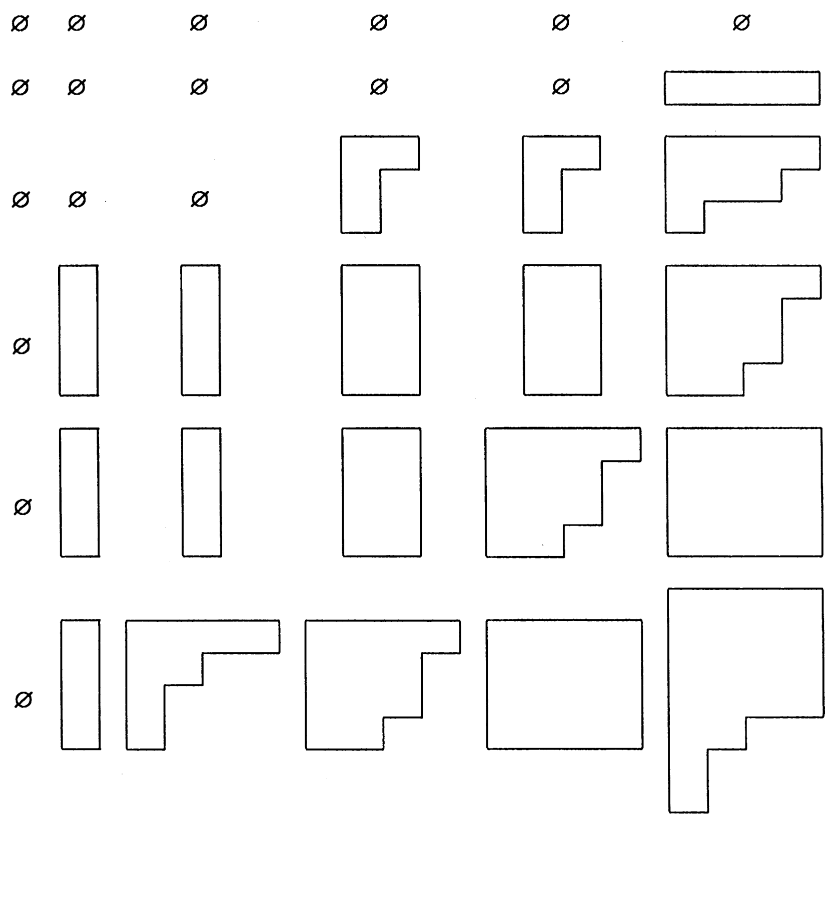
Proposition 20. *The rules* (C1)–(C6) *yield a well-defined map $\Phi:\mathscr{B}\to\mathscr{A}$ that is the inverse of Ψ.*

As mentioned before, our inverse k-correspondence defined by (I1)–(I6) induces a k-ribbon Schensted bijection by [4, Theorem 3.6.1]. This particular k-ribbon Schensted bijection shall be called the color-to-spin k-ribbon Schensted bijection, since it satisfies (2.2) due to (6.7).

Theorem 4 follows from the fact that taking the inverse of π, corresponds to transposing the matrices $(c_{i,j})$ and $(\tau_{i,j})$, and that the rule Ψ for computing in the two-by-two matrices, is transpose-invariant by (6.8).

As in the proof of [4, Theorem 3.6.1], the inverse of the map $\pi \to (P(\pi), Q(\pi))$ may be described in terms of the inverse k-correspondence Ψ or $\Phi = \Psi^{-1}$. Let P and Q be k-ribbon tableaux of the same shape. Initialize the nth row (resp. nth column) of the matrix $(\tau_{i,j})_{0 \leqslant i,j \leqslant n}$ by the chains of shapes given by P and Q respectively. One then uses Φ to compute the rest of the matrix, working from southeast to northwest, by the rule $\Phi(\tau_{i+1,j}, \tau_{i,j+1}, \tau_{i+1,j+1}) = (\tau_{i,j}, \tau_{i+1,j}, \tau_{i,j+1}, c_{i,j})$, which also computes the matrix $(c_{i,j})$. This recovers in a well-defined way the matrix c_{ij}. Moreover this matrix defines a colored permutation π. Since $\Phi = \Psi^{-1}$ it follows that $P = P(\pi)$ and $Q = Q(\pi)$.

Example 21. The matrix for the colored indexed permutation π of Example 12 is given below.



7. Remarks

7.1. Cores

The above constructions can be adjusted easily for the presence of a k-core κ. The only difference in the definitions is that instead of filling the zeroth row and column of the table $\tau_{i,j}$ with the empty partition, κ is used in its place.

7.2. Column insertion version

We have defined a "row insertion" color-to-spin algorithm. A "column insertion" version is obtained if the words "northeast" and "southwest" are exchanged, and $\leqslant$ is replaced by $\geqslant$ in Lemma 6. If colors are complemented, that is, c is replaced by $k-c$, the resulting tableau pair is transposed.

7.3. Comparison with Stanton–White Rim Hook Schensted

As is evident from the definition of inverse k-correspondence, the standard case of these two algorithms have a very similar intrinsic structure, despite their considerable differences mentioned in the introduction.

Define the *orientation* of a cell (i,j) by $\mathrm{o}(i,j)=\mathrm{diag}(i,j) \bmod k=(j-i) \bmod k$, the orientation of a k-ribbon to be the orientation of its head, and the orientation of a k-ribbon tableau to be the sum of the orientations of its k-ribbons.

The Stanton–White k-ribbon bijection, and the color-to-spin k-ribbon bijection defined here, differ only in the following way: if everywhere in the definitions the word "spin" is replaced by "orientation", then one obtains the Stanton–White algorithm! This only requires modifying (I5) and (I6). So perhaps the two k-ribbon bijections should be referred to as the color-to-orientation and the color-to-spin bijections. As mentioned before, this striking similarity evaporates altogether in the semistandard case.

7.4. Garfinkle's domino Schensted

In the domino $(k=2)$ case, Garfinkle's domino insertion also has the color-to-spin property [11]. Garfinkle's domino insertion and the $k=2$ case of the color-to-spin Schensted differ in the following ways.

Suppose a domino of color c of maximum value is inserted. In Garfinkle's insertion, this domino is adjoined at the southwestmost position of spin 1 if $c=1$ and at the northeastmost position of spin 0 if $c=0$. In the domino case of our insertion, the domino is always adjoined at the northeastmost position of spin c.

The only other case where these two domino insertions differ, is when $h_1=h_2\neq\emptyset$. In Garfinkle's insertion, the dominoes of spin 0 and 1 are bumped to the southwest and northeast, respectively, while in our insertion all dominoes are bumped to the southwest. In both of these domino insertions, all bumped dominoes retain their spin.

In other words, Garfinkle's insertion uses row insertion of dominoes of spin 0 and column insertion of dominoes of spin 1, whereas ours always uses row insertion.

Appendix A.

For the proofs of Lemmas 6 and 14 it is convenient to employ an encoding of a partition μ by a doubly infinite bit sequence $(\dots, b_{-2}, b_{-1}, b_0, b_1, b_2, \dots)$ that is called the *edge sequence* of μ in [16]. One imagines that as the index i of b_i decreases, the southeast boundary of the Ferrers diagram of μ is being traced from northeast to southwest. A westward unit segment is encoded by a 0 and a southward unit segment is encoded by a 1. By convention the unique lattice point on the main diagonal touched by the boundary of the partition, falls between the segments indexed by $-1, 0 \in \mathbb{Z}$. So for $i \gg 0$ $b_i = 0$ and for $i \ll 0$, $b_i = 1$. If the partition must be emphasized then the edge sequence of μ is denoted b^μ. For example, the partition $(4, 3, 1, 0, 0, \dots)$ has the following edge sequence:

i	$\dots$	-4	-3	-2	-1	0	1	2	3	4	$\dots$
b_i	1	1	0	1	0	0	1	0	1	0	0

Suppose $h = \lambda/\mu$ is a k-ribbon with head in diagonal i. Then $b^\mu_i = 0$, $b^\mu_{i-k} = 1$, and b^λ is obtained from b^μ by exchanging these two values. Also $\mathrm{sp}(h) = \sum_{j=i-k+1}^{i-1} b^\mu_j = \sum_{j=i-k+1}^{i-1} b^\lambda_j$, the number of 1's strictly between the two bits being flipped. Let $S_\mu(d) = \sum_{i=d-k+1}^{d} b^\mu_i$. Then

$$S_\mu(d) = 0 \quad \text{for } d \gg 0, \tag{8.1}$$

$$S_\mu(d) = k \quad \text{for } d \ll 0, \tag{8.2}$$

$$S_\mu(d-1) - S_\mu(d) = b^\mu_{d-k} - b^\mu_d \in \{-1, 0, 1\}. \tag{8.3}$$

Remark A.1. From (8.3), it is seen that for a fixed color c,

1. $S_\mu(d) = c$ and $S_\mu(d-1) = c+1$ if and only if there is a μ-addable k-ribbon of spin c with head in diagonal d.
2. $S_\mu(d) = c+1$ and $S_\mu(d-1) = c$ if and only if there is a μ-removable k-ribbon of spin c with head in diagonal d.

Let λ/μ be a k-ribbon with head in diagonal i. Then

$$S_\lambda(d) - S_\mu(d) = \begin{cases} 1 & \text{if } i \leqslant d < i+k, \\ -1 & \text{if } i-k \leqslant d < i, \\ 0 & \text{otherwise.} \end{cases} \tag{8.4}$$

Proof (*of Lemma* 6). For point 1, observe that as d decreases from large to small values, $S_\mu(d)$ starts at 0 by (8.1), changes by $-1, 0, 1$ as d is decremented by (8.3), and stabilizes at $k > c$ by (8.2). Clearly for some d, $S_\mu(d) = c$ and $S_\mu(d-1) = c+1$. Then Remark A.1 point 1 applies. For point 2, let $d_0 = \mathrm{diag}(\mathrm{hd}(h))$. Since h is μ-removable, by Remark A.1 point 2, $S_\mu(d_0 - 1) = \mathrm{sp}(h) \leqslant c$. Starting at $d = d_0 - 1$ and decrementing d, by (8.2) and (8.3) there is an index $d < d_0$ such that $S_\mu(d) = c$ and $S_\mu(d-1) = c+1$. Again Remark A.1 point 1 applies. $\square$

Proof (*of Lemma* 14). (B1) follows immediately by Lemma 6 with $p = 0$. For the part of (B3) pertaining to this case, consider the construction of p and h'. Suppose $0 \leqslant p < m$. Write $d_j = \mathrm{diag}(\mathrm{hd}(h_j))$ for $1 \leqslant j \leqslant m$ and $d' = \mathrm{diag}(\mathrm{hd}(h'))$. It must be shown that $d' < d_{p+1}$.

Suppose first that $d' = d_{p+1}$. Then h' and h_{p+1} agree since both are $\lambda^{(p)}$-addable. In particular, $\mathrm{sp}(h_{p+1}) = c$. Since h_{p+1} is $\lambda^{(p+1)}$-removable, by Remark A.1 point 2 $S_{\lambda^{(p+1)}}(d_{p+1}) = c + 1$. Arguing as above there is a $\lambda^{(p+1)}$-addable ribbon h'' of spin c such that $h_{p+1} <_d h''$. Then $h_{p+1} <_c h''$, contradicting the maximality of p.

Suppose next that $d_{p+1} < d'$. By (8.4), $c + 1 = S_{\lambda^{(p)}}(d' - 1) \leqslant S_{\lambda^{(p+1)}}(d' - 1)$, with equality only if $d' - 1 \geqslant d_{p+1} + k$. If equality does not hold, then $S_{\lambda^{(p+1)}}(d') \geqslant c+1$. Then there is a $\lambda^{(p+1)}$-addable ribbon h'' of spin c with $h_{p+1} <_d h' <_d h''$. Again $h_{p+1} <_c h''$, contradicting the maximality of p. If equality holds then $S_{\lambda^{(p+1)}}(d' - 1) = c + 1$ and by (8.4) $S_{\lambda^{(p+1)}}(d') = S_{\lambda^{(p)}}(d') = c$. In other words, there is an $\lambda^{(p+1)}$-addable ribbon h'' of spin c with head in diagonal d'. As before $h_{p+1} <_d h''$ and $h_{p+1} <_c h''$, contradicting the maximality of p. Therefore $d' < d_{p+1}$ as desired.

For (B2), if $m = 0$ then $p = 0$ and h' exists by Lemma 6 and (B3) is trivial. Suppose $m > 0$ and $h_m <_d h$. Let $d^* = \mathrm{diag}(\mathrm{hd}(h))$. By Remark A.1 $S_{\lambda^{(m)}}(d^* - 1) = \mathrm{sp}(h) \leqslant c$. If $S_{\lambda^{(m)}}(d - 1) > c$ for some $d_m < d \leqslant d^* - 1$ then there is a λ-addable ribbon h' of spin c with $h_m <_d h' <_d h$. As before $h_m <_c h'$ and the desired h' exists. Again (B3) is vacuously true. Otherwise $S_{\lambda^{(m)}}(d) \leqslant c$ for all $d_m \leqslant d < d^* - 1$. In particular $S_{\lambda^{(m)}}(d_m) = \mathrm{sp}(h_m) + 1 \leqslant c$ so that $\mathrm{sp}(h_m) < c$. Consider the $\lambda^{(m-1)}$-removable ribbon $\hat{h} = \mathbf{bumpin}(h, h_m)$, which has $\mathrm{hd}(\hat{h}) = \mathrm{hd}(h)$ since $h_m <_d h$. By induction there is a p and h' such that $h_p <_c h' <_d h$ and if $p + 1 \leqslant m - 1$ then $h' <_d h_{p+1}$. This is enough unless $p + 1 = m$ in which case it must be shown that $h' <_d h_m$. By assumption $h_{m-1} <_c h' <_d h$. By Remark A.1 point 1, $S_{\lambda^{(m-1)}}(d' - 1) = c + 1$. It follows that $d' - 1 < d_m$. Since $\mathrm{sp}(h_m) < c$, one has $d' < d_m$ as desired. $\square$

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